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Recognizing random intersection graphs

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Abstract

A polynomial-time algorithm is given which succeeds in reconstructing the simple k-uniform hypergraph H from its ℓ-intersection graph, for almost all random k-uniform hypergraphs $H = \mathscr{H}_k(n, p)$, where $p \succ n^{-1/2+\varepsilon}$, $\varepsilon > 0$. Two related algorithms reconstruct almost every random graph $G = \mathscr{G}(n, p)$ from its k-line graph $L_k(G)$ (which is the $(k-1)$-intersection graph of the set of all complete subgraphs on k vertices), and almost every random graph G from its $(k-1)$-in-k graph $\Phi_{k-1,k}(G)$ (which has all complete $(k-1)$-vertex subgraphs of G as vertices, two of them adjacent if they lie in some common complete k-vertex subgraph), for $p \succ n^{-1/k+\varepsilon}$, respectively, $p \succ n^{-1/(2k-2)+\varepsilon}$, $\varepsilon > 0$.

1. Introduction

The *ℓ-intersection graph* $\Omega_\ell(H)$ of a simple hypergraph $H = (A, \{S_x / x \in V\})$ has V as vertex set and two distinct vertices $x \neq y$ are joined by an edge whenever $|S_x \cap S_y| \geqslant \ell$. Except the well-known case $\ell = 1$, $k = 2$ of line graphs, the question of how to recognize ℓ-intersection graphs of k-uniform hypergraphs for $1 \leqslant \ell < k$ seems a difficult one; it is NP-complete for $\ell = 1$ and $k \geqslant 3$ [13], as well as for $2 \leqslant \ell = k - 1$ [9].

On the other hand, characterizations of $(k-1)$-intersection graphs of certain k-uniform hypergraphs were given in the late sixties and early 1970s. A characterization of 2-intersection graphs of *complete* 3-uniform hypergraphs with more than 16 [5] or less than 9 [2] vertices was generalized to a characterization of $(k-1)$-intersection graphs of complete k-uniform hypergraphs with more than $2k(k-1)+4$ vertices in [7]. Let $H(n_1, n_2, \ldots, n_t; t, k)$ be the k-uniform hypergraph whose vertex set is the disjoint union of t sets $V_1, V_1, \ldots, V_t$, $|V_i| = n_i$, and whose hyperedges are all k-sets S where $|S \cap V_i| \leqslant 1$ for every $1 \leqslant i \leqslant t$. Then in [10] a characterization of 2-intersection graphs

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of hypergraphs $H(n,n,n;3,3)$ was given for $n>7$, and it was shown in [1] that it also holds for every integer $n \neq 4$. More general, a characterization of $(k-1)$-intersection graphs of hypergraphs $H(n,\dots,n;t,k)$ was given for $k \leqslant 5$ and some additional conditions in [15]. See also [6] and [3] for characterizations of $(k-1)$-intersection graphs of certain $H(n_1,\dots,n_k;k,k)$, and [4], [8] for more information.

These characterizations were certainly inspired and motivated by the regularity of the graphs and hypergraphs considered, but from an algorithmical point of view it turns out that not regularity but a certain 'denseness' of the hypergraphs is the reason why the graphs can be recognized in polynomial time. The rather natural method of how to recognize $(k-1)$-intersection graphs of 'dense' k-uniform hypergraphs is presented in Sections 2 and 3. Then these results can be used for recognizing ℓ-intersection graphs of 'dense' k-uniform hypergraphs, see Sections 5 and 6. Although the concepts of 'denseness' used vary, it turns out that almost every random k-uniform hypergraph has all these densenss properties, therefore the algorithms presented work in almost all cases.

Sections 4 and 7 deal with some related concepts, recognizing k-line graphs or ℓ-in-k graphs of graphs, which can also be dealt with in a quite similiar way.

We use the usual model for random k-uniform hypergraphs. Let n be a positive integer, and $p=p(n)$ be any edge probability, We construct a random k-uniform hypergraph $\mathscr{H}_k(n,p)$ by taking $\{1,2,\dots,n\}$ as vertex set, and including every k-element subset independently with probability p to our hyperedge set. For $k=2$, this is just the standard model $\mathscr{G}(n,p)$ for random graphs.

What we need in Sections 4 and 7 are not random hypergraphs, but rather uniform hypergraphs derived from random graphs. For integers $k \geqslant 3$, the *K_k-hypergraph $H^k(G)$* of a graph G has the same vertex set, and all K_ks as hyperedges — certainly it is k-uniform. One could view a graph as its own K_2-hypergraph.

For a hypergraph H, let $A(H)$ denote its vertex set.

Throughout the paper we use the following asymptotics:

Lemma 1. *For every fixed integers k,t and every fixed $\varepsilon>0$:*

$$n^k\left(1-\frac{n^{\varepsilon}}{n}\right)^{n-t} \to 0 \quad \text{as } n\to\infty.$$

Proof. Since for every fixed $c \geqslant 0$ the sequence $(1-c/n)^n$ is increasing for $n \geqslant c$, and converges towards e^{-c}, we get

$$0 \leqslant \left(1-\frac{n^{\varepsilon}}{n}\right)^{n} < \mathrm{e}^{-n^{\varepsilon}} \quad \text{for } n \geqslant n^{\varepsilon}.$$

Applying several times the Theorem of Bernoulli and l'Hospital, we obtain that $n^k \mathrm{e}^{-n^{\varepsilon}} \to 0$ as $n\to\infty$. Finally, note that $(1-n^{\varepsilon}/n)^t \to 1$ as $n\to\infty$, for every fixed t. $\square$

2. $(k-1)$-set hypergraphs of k-uniform hypergraphs

Let $(A, \{S_x/x \in V\})$ be a k-uniform hypergraph. For $1 \leqslant \ell < k$, its *ℓ-set hypergraph* $H[\ell]$ has all ℓ-element subsets of hyperedges of H as vertices. For $x \in V$, define $S_x[\ell] = \{A_1, A_2, \ldots, A_{\binom{k}{\ell}}\}$, where $A_1, A_2, \ldots, A_{\binom{k}{\ell}}$ are all ℓ-element subsets of S_x. The family of hyperedges of $H[\ell]$ is just $(S_x[\ell]/x \in V)$. The *ℓ-skeleton* $\mathrm{Sk}_\ell(H)$ of a hypergraph H is the ℓ-uniform hypergraph with the same vertices than H, and all ℓ-element subsets of hyperedges of H as hyperedges.

In the remainder of the section we assume that H is simple (i.e. all hyperedges are distinct sets) and k-uniform, and $\ell = k-1$. Then $H[k-1]$ must be linear (i.e. every two distinct hyperedges contain at most one common vertex) and also simple and k-uniform.

A natural approach to test whether a given simple, linear k-uniform hypergraph H' is the $(k-1)$-set hypergraph of some simple k-uniform hypergraph H would be the following: Assume $H' = H[k-1]$. Under this assumption, in the first step we try to find out which pairs of vertices of H' have — viewed as $k-1$-element subsets of $A(H)$ — $k-2$ common elements. These pairs are sampled as edges of the graph $F = (A(H'), E)$. If this has been done, we only need to check in the second step whether the resulting graph is a $(k-2)$-intersection graph of some simple $(k-1)$-uniform hypergraph H''. Thereby, H'' should be the $(k-1)$-skeleton of some k-uniform hypergraph, i.e. it should obey the following property: For every hyperedge S_0 of H'' there should be $k-1$ further hyperedges $S_1, \ldots, S_{k-1}$ such that $|\bigcup_{i=0}^{k-1} S_i| = k$. This second step sounds difficult enough, but for $k=3$ we can do it, since we can recognize line graphs.

If two distinct $(k-1)$-element subsets are both contained in the same k-element superset, then they have exactly $k-2$ common elements. Thus,

(1) all pairs of vertices lying in the same hyperedge of H' are included as edges in F.

However, we may have to include more. Assume that the vertices x and y of H' are in no common hyperedge, but that another vertex z joins a common hyperedge with x and another common hyperedge with y. If X, Y, and Z are the corresponding $(k-1)$-element subsets of $A(H)$, then $|X \cap Z| = k-2$ and $|Y \cap Z| = k-2$, thus $|X \cap Y| \geqslant k-3$. If $|X \cap Y| = k-3$, at most four such vertices z are possible. Thus, if there are at least five such vertices z, then we know $|X \cap Y| = k-2$ and add xy to our edge set E.

(2) If there are distinct vertices $x, y, z_1, z_2, z_3, z_4, z_5$ in H' where for every $1 \leqslant i \leqslant 5$ some hyperedge of H' contains both x and z_i, and another hyperedge contains both y and z_i, then we add xy as edge of F.

There may be, however, still more edges in F. Not so if H has the following property, see the following proposition.

Definition 2. A k-uniform hypergraph obeys property A if for every two $(k-1)$-element subsets S and T where $|S\cap T|=k-2$ either $S\cup T$ is a hyperedge, or there are at least 5 elements $a_1,\dots,a_5$ outside $S\cup T$ such that all $S\cup\{a_i\}$, $T\cup\{a_i\}$ are hyperedges.

Proposition 3. *Let $H'=H[k-1]$ for the k-uniform hypergraph H with property A be given. Then the graph F constructed by the two rules* (1) *and* (2) *above is the* $(k-2)$*-intersection graph of the* $(k-1)$*-skeleton* $\mathrm{Sk}_{k-1}(H)$ *of* H.

Proof. Let x,y be vertices of H'. The preceding discussion showed that if $xy\in E(F)$, then $|X\cap Y|=k-2$ for the corresponding $(k-1)$-element subsets X,Y of the vertex set of H. Assume conversely $|X\cap Y|=k-2$ but (1) cannot be applied since x and y are in no common hyperedge in H'. Then, $X\cup Y$ is no hyperedge of H, therefore by property A there are $a_1,a_2,\dots,a_5\in A(H)\setminus(X\cup Y)$ with all $X_i=X\cup\{a_i\}$, $Y_i=Y\cup\{a_i\}$, $1\leqslant i\leqslant 5$, hyperedges in H. Then, the sets $(X\cap Y)\cup\{a_i\}, 1\leqslant i\leqslant 5$, correspond to a vertices $z_i, 1\leqslant i\leqslant 5$ of H'. Actually, these vertices obey (2) above, thus $xy\in E(F)$. □

The hypergraphs $\binom{(1,2,\dots,n)}{k}$ have property A for $n\geqslant k+5$. More general, the hypergraphs $H(n_1,\dots,n_t;t,k)$, where $n_1\geqslant n_2\geqslant\cdots\geqslant n_t$, have property A if $\sum_{i=k}^{t} n_i\geqslant 5$. Another example are random $H=\mathscr{H}_k(n,p)$, with large enough n and high enough edge probability p.

Proposition 4. *Let $p=p(n)\succ n^{-1/2+\varepsilon}$, for $\varepsilon>0$. Then, almost surely, $\mathscr{H}_k(n,p)$ obeys property A.*

Proof. What is the probability that given two $(k-1)$-element subsets X and Y with $k-2$ common elements, there are no $a_1,a_2,\dots,a_5$ as in the definition of property A? Of the $n-k$ independent events that both $X\cup\{z\}$ and $Y\cup\{z\}$ are hyperedges of $\mathscr{H}_k(n,p)$, for $z\notin X\cup Y$, having each probability p^2, at most four should occur. The probability for that is

$$\begin{aligned}&(1-p^2)^{n-k}+(n-k)(1-p^2)^{n-k-1}p^2+\cdots+\binom{n-k}{4}(1-p^2)^{n-k-4}p^8\\&\quad=(1-p^2)^{n-k-4}\mathrm{O}(n^4)\end{aligned}$$

There are at most $\binom{k}{2}\binom{n}{k}$ such pairs X,Y of $(k-1)$-element sets with $k-2$ common vertices. The events that X and Y behave as required in property A are not independent, but the probability that there is some pair (X,Y) that does *not* behave in this way is at most

$$\binom{k}{2}\binom{n}{k}(1-p^2)^{n-k-4}\mathrm{O}(n^4)\leqslant\mathrm{O}(n^{k+4})\left(1-\frac{n^{2\varepsilon}}{n}\right)^{n-k-4}\to 0\quad\text{for } n\to\infty$$

by Lemma 1. □

Property A is heriditary to skeletons:

Lemma 5. *If the k-uniform hypergraph H has property A, then* $\mathrm{Sk}_{k-1}(H)$ *has too.*

Proof. Choose any two such $(k-2)$-element subsets S and T with $|S \cap T| = k-3$. Choose any element $a \notin S \cup T$. Since H has property A, either $S \cup T \cup \{a\}$ is a hyperedge of H, i.e. $S \cup T$ a hyperedge of $\mathrm{Sk}_{k-1}(H)$, or we find $a_1, \dots, a_5$ such that for each $1 \leqslant i \leqslant 5$, $S \cup \{a, a_i\}$ and $T \cup \{a, a_i\}$ are hyperedges of H. But then all of $S \cup \{a_i\}$, $T \cup \{a_i\}$, $1 \leqslant i \leqslant 5$, are hyperedges of $\mathrm{Sk}_{k-1}(H)$. □

3. *k*-facet graphs

We call $(k-1)$-intersection graphs of simple k-uniform hypergraphs *k-facet graphs*. Cliques in k-facet graphs come in two types, called star cliques and hole cliques in [11]. Either the hyperedges all join a common $(k-1)$-element subset — the star clique case — , or they are all contained in some $(k+1)$-element superset — the hole clique case. Both is possible, but only for cliques with exactly two vertices; these cliques are called star cliques by convention. Every clique must have one of these two types.

Every $(k-1)$-element S subset of any hyperedge in a k-uniform hypergraph H generates a complete graph S^* in $\Omega_{k-1}(H)$ — we put all vertices x in S^* where $S \subseteq S_x$. We call these graphs S^* *star graphs*. Certainly every star clique is a star graph, but there may be star graphs that are not cliques. These star graphs have 1 or 2 vertices. Note that the star graphs form a partition of the edge set of $\Omega_{k-1}(H)$ and each vertex is covered exactly k times. Therefore, the 1- and 2-vertex star graphs are uniquely determined by the star cliques.

The standard approach would be to find out which cliques are star cliques, then find out all additional star graphs, and after that, test whether the dual of the hypergraph with the star graphs as hyperedges is the $(k-1)$-set hypergraph $H[k-1]$ of some k-uniform hypergraph H.

Unfortunately, these star cliques are not uniquely recognizable in the k-facet graph.

There are some simple properties, however: We know that hole cliques contain at least 3 and at most $k+1$ vertices.

Definition 6. A k-uniform hypergraph has property B if every $(k-1)$-element subset of every hyperedge lies in more than $k+1$ hyperedges of H.

For k-facet graphs of k-uniform hypergraphs with property B, this ambiguity of which cliques are star cliques vanishes, since all star cliques contain more than $k+1$ vertices.

Complete k-uniform hypergraphs $\binom{\{1,2,\dots,n\}}{k}$ have property B for $n > 2k$. More general, the hypergraphs $H(n_1, \dots, n_t; t, k)$, where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t$, have property B if $\sum_{i=k}^{t} n_i \geqslant k+1$. Again, certain random hypergraphs have this property:

Proposition 7. *If* $p \succ n^{-1+\varepsilon}$, $\varepsilon > 0$, *then almost every* $\mathscr{H}_k(n,p)$ *has property B.*

Proof. Let S be a $(k-1)$-element subset of the vertex set of $\mathscr{H}_k(n,p)$. For every single point a outside S, the probability that $S\cup\{a\}$ is a hyperedge equals p. Therefore, the probability that there are at most $k+1$ such points a for which $S\cup\{a\}$ is a hyperedge equals

$$(1-p)^{n-k+1}+(n-k+1)(1-p)^{n-k}p+\binom{n-k+1}{2}(1-p)^{n-k-1}p^2$$
$$+\cdots+\binom{n-k+1}{k+1}(1-p)^{n-2k}p^{k+1}=(1-p)^{n-2k}\mathrm{O}(n^{k+1}).$$

The probability for the existence of such a bad S is therefore at most $\binom{n}{k-1}(1-p)^{n-2k}\mathrm{O}(n^{k+1})$, which goes to 0 as $n\to\infty$ by Lemma 1. □

Again property B is heriditary to skeletons:

Lemma 8. *If the k-uniform hypergraph H has property B, then* $\mathrm{Sk}_{k-1}(H)$ *has too.*

Proof. Take any $(k-2)$-element subset $S=Y\setminus\{a\}$ of any hyperedge Y of $\mathrm{Sk}_{k-1}(H)$. Note that Y is a subset of some hyperedge in H. Since H has property B, there are at least $k+2$ hyperedges $X_1,X_2,\ldots,X_{k+2}$ of H containing Y. But now $X_1\setminus\{a\}$, $X_2\setminus\{a\},\ldots,X_{k+2}\setminus\{a\}$ are hyperedges of $\mathrm{Sk}_{k-1}(H)$ containing S. □

Theorem 9. *For every $k\geqslant 2$, there is a polynomial-time algorithm that decides whether a given graph is the $(k-1)$-intersection graph of some k-uniform hypergraph H having both properties A and B. If it is so, H is unique.*

Proof. The algorithm is recursive and goes as follows. Let n be the number of vertices of our graph G. It is known that all k-facet graphs are (K_5-e)-free [11]; this we have to check first (in time $\mathrm{O}(n^5)$). Next, we compute all cliques of size larger than $k+1$ — these are supposed to be the star cliques. Since the graph is (K_5-e)-free, this can be done simply by computing all K_4, and extending them to cliques by checking all further vertices whether it is adjacent or not to all four vertices in question. Again we need time $\mathrm{O}(n^5)$. Note that since the $(k-1)$-skeleton of a k-uniform hypergraph with n hyperedges has at most kn hyperedges, and since every star clique stems from such a hyperedge, there should be no more than kn star cliques. Next, we consider the dual of the hypergraph formed by these star cliques as hyperedges, this dual should be the $(k-1)$-set hypergraph of our original hypergraph H. To test this, we perform steps (1) and (2) above to construct F, which should be the $(k-2)$-intersection graph of the $(k-1)$-skeleton of H. Note that F has at most $kn=\mathrm{O}(n)$ vertices. Checking whether the hypergraph H' obtained is a $(k-1)$-skeleton, and obtaining H from H' can be done in time $\mathrm{O}(nm)$ where n and m are the numbers of vertices and hyperedges of H'. But H' should contain at most kn vertices and at most n hyperedges (otherwise we may stop), so we get time $\mathrm{O}(n^2)$. When we eventually arrive at testing intersection graphs of 2-uniform hypergraphs [12], [16] we are done. Note that we never loose uniqueness during the procedure, therefore there is at most one such hypergraph H. The overall running time, for fixed k, is $O(n^5)$. □

This algorithm can be adapted for a general k-facet graph test. If the graph G contains some induced $K_5 - e$, then it is definitely no k-facet graph. If G is $(K_5 - e)$-free and the above approach yields some simple k-uniform hypergraph H, then we know definitely that G is a k-facet graph. The only problem arises if G is $(K_5 - e)$-free, but the approach above fails (since G is not the k-facet graph of some simple k-uniform hypergraph having both properties A and B) — in this case this variant of the algorithm outputs a '?', indicating that it cannot settle the question. So we are dealing with algorithms with three possible outcomes: Either it claims that the graph G is no k-facet graph, and gives a proof by showing some induced $K_5 - e$, or it claims that G is the k-facet graph of some simple k-uniform hypergraph H, and presents H as proof, or it says that it is not sure.

Such an algorithm should have the property that this third case, where it cannot decide, should occur seldom. Here we need a distribution. The obvious one, testing the algorithm on random graphs, is not satisfactory, since k-facet graphs are so rare — almost no graph is a k-facet graph. Therefore, with this notion of 'seldom', even the trivial algorithm which rejects graphs with induced $K_5 - e$, but accepts none, would work in almost all cases. Therefore, let us assume that our input graph G is created in the following way. First a coin is flipped. If it shows head, a random graph $\mathscr{G}(n, p)$ is taken as G. If it shows tail, a random simple k-uniform hypergraph $\mathscr{H}_k(n, p))$ is chosen, and its k-facet graph is taken as G. It follows that the algorithm presented works for almost all cases, as $n \to \infty$.

Corollary 10. *Let $p = p(n) \succ n^{-1/2+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, $\Omega_{k-1}\mathscr{H}_k(n, p)$ can be recognized.*

4. *k*-line graphs

We define the *k-line graph $L_k(G)$* of a graph G as $\Omega_{k-1}(H^k(G))$. It is somewhat easier to deal with k-line graphs than k-facet graphs, since the star cliques are revealed immediately; at least there are only two candidate sets for star cliques. Nevertheless, we will not pursue this further, since for random graphs we gain nothing.

Theorem 11. *Let $k \geqslant 3$, and $p = p(n) \succ n^{-1/k+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, $L_k\mathscr{G}(n, p)$ can be recognized.*

Proof. $L_k\mathscr{G}(n, p) = \Omega_{k-1}(H^k(\mathscr{G}(n, p)))$, but $H^k(\mathscr{G}(n, p))$ has almost surely both Properties A and B. The proof for property A is almost identical to the counting in the proof of Proposition 4. For property B, take any complete $(k-1)$-vertex subgraph S of $\mathscr{G}(n, p)$. Property B is violated at S if it can be extended in fewer than

$k+2$ ways to some complete graph on k vertices. The probability for this event is

$$\sum_{i=0}^{k+1}\binom{n-k+1}{i}(1-p^{k-1})^{n-k+1-i}p^{(k-1)i}=(1-p^{k-1})^{n-2k}\mathrm{O}(n^{k+1}).$$

We have at most $\binom{n}{k-1}$ possibilities for such a set S, therefore the probability that property B is violated at any of them is at most $(1-p^{k-1})^{n-2k}\mathrm{O}(n^{(k+1)(k-1)})$, which again converges to 0 as $n\to\infty$ by Lemma 1. □

5. Intersection graphs of k-uniform hypergraphs

In addition to properties A and B, our hypergraph should now obey the following:

Definition 12. A k-uniform hypergraph obeys property C if for every set T of at most $2k$ points, and every point $a\notin T$ there is some hyperedge containing a and disjoint from T.

Again large complete hypergraphs $\binom{\{1,2,\ldots,n\}}{k}$ where $n\geqslant 3k$ obey property C. Also certain random hypergraphs have it:

Proposition 13. *Let $p=p(n)\succ n^{-(k-1)+\varepsilon}$, for $\varepsilon>0$. Then, almost surely, $\mathscr{H}_k(n,p)$ obeys property C.*

Proof. Assume $a\notin T$, where $a\in A$, $T\subseteq A$, $|T|\leqslant 2k$. For every $(k-1)$-element subset S of $A\setminus(T\cup\{a\})$, $S\cup\{a\}$ is the hyperedge required in property C with probability p. There are $\binom{|A|-|T|-1}{k-1}$ such sets S, and all these events are independent, therefore the probability that none of them is a hyperedge is at most $(1-p)^{\binom{|A|-|T|-1}{k-1}}\approx(1-p)^{\binom{n}{k-1}}\to 0$, as $n\to\infty$. □

Having this property, we can reduce the problem of recognizing intersection graphs of k-uniform hypergraphs to that of recognizing k-facet graphs.

Assume $G=\Omega(H)$, where H is k-uniform and obeys property C. Let x and y be adjacent vertices in G, corresponding to the hyperedges X and Y of H. If $|X\cap Y|=k-1$, then every hyperedge intersecting X but not Y must contain the unique point of $X\setminus Y$. Thus, all these hyperedges are pairwise intersecting, therefore no induced $K_{1,3}$ in G contains both x and y, one of them as center. On the other hand, if $|X\cap Y|\leqslant k-2$, then $X\setminus Y$ contains at least two elements a_1,a_2. By property C, there is some hyperedge X_1, disjoint from $Y\cup\{a_2\}$ but containing a_1. Again by property C there is another hyperedge X_2 disjoint from $Y\cup X_1$ and containing a_2. Therefore, x_1,x_2,x,y form an induced $K_{1,3}$ with center x.

Therefore, whether or not $|X \cap Y| = k - 1$ for adjacent vertices x, y – intersecting hyperedges X, Y – is revealed, and we can construct a graph J as follows:

(3) Adjacent vertices x, y of G are adjacent in J if and only if there is no induced $K_{1,3}$ in G that contains both x and y, and one of them as its center.

Theorem 14. *For every $k \geqslant 2$, there is a polynomial-time algorithm that decides whether a given graph is the intersection graph of some k-uniform hypergraph H having all properties A, B, and C. If it is so, H is unique.*

Proof. We construct the graph J by rule (3). Then we test whether or not J is the facet graph of some k-uniform hypergraph obeying properties A and B, as described in Theorem 9. For the construction (3) we need time $O(nm)$, which is dominated by the time $O(n^5)$ to test k-facet graphs. □

Corollary 15. *Let $p = p(n) \succ n^{-1/2+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, $\Omega(\mathscr{H}_k(n, p))$ can be recognized in polynomial time.*

6. ℓ-intersection graphs of k-uniform hypergraphs

The approach in Section 5 can be generalized to recognize almost all ℓ-intersection graphs of k-uniform hypergraphs, but now the running time of the algorithms increases with ℓ, k. First we generalize property C:

Definition 16. For integers $\ell < k$ and t, a k-uniform hypergraph obeys property $C(\ell, t)$ if for every ℓ-element subsets S and every set T, $|T| \leqslant t$, disjoint from S, there is some hyperedge X containing S but disjoint from T.

Property C above is just Property $C(1, 2k)$.

Proposition 17. *Let $\ell < k$ and t be fixed integers. Let $p = p(n) \succ n^{-(k-1)+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, $\mathscr{H}_k(n, p)$ obeys property $C(\ell, t)$.*

Proof. The proof is essentially identical to the proof of Proposition 13. □

As in Section 5, we can reduce the problem of recognizing ℓ-intersection graphs of k-uniform hypergraphs H to the problem of recognizing k-facet graphs if H has property $C(\ell, \binom{k-1}{\ell-1}(k-\ell) + 2k - \ell)$. Let x and y be adjacent in $G = \Omega_\ell(H)$. If $|X \cap Y| = k - 1$, then x has in G at most $\binom{k-1}{\ell-1}$ pairwise nonadjacent neighbors which are not adjacent to y. On the other hand, if $|X \cap Y| \leqslant k - 2$, then let $S_1, S_2, \ldots, S_p$ be all ℓ-element subsets of X which are not totally contained in Y. Obviously $p = \binom{k}{\ell} - \binom{|X\cap Y|}{\ell} > \binom{k-1}{\ell-1}$, since $\binom{|X\cap Y|}{\ell} < \binom{k-1}{\ell}$. Thus $t = \binom{k-1}{\ell-1} + 1 \leqslant p$. Since, we have Property $C(\ell, \binom{k-1}{\ell-1}(k-\ell) + 2k - \ell)$, there is some hyperedge X_1 with $S_1 \subseteq X_1$ and

$X_1 \cap (X \cup Y \setminus S_1) = \emptyset$. In the same way, there is some hyperedge X_2 including S_2 and obeying $X_2 \cap (X \cup Y \cup X_1 \setminus S_2) = \emptyset$. We proceed and find hyperedges $X_1, X_2, \dots, X_t$. Note that each pair of hyperedges out of $Y, X_1, X_2, \dots, X_t$ has fewer than ℓ common vertices by the construction, but all of them intersect X into some ℓ-element set. Therefore, $x, y, x_1, x_2, \dots, x_t$ forms an induced $K_{1,t+1}$ with center x.

We construct again a graph which should be the k-facet graph of our hypergraph.

(4) Adjacent vertices x, y of G are adjacent in $J(\ell, k)$ if and only if there is no induced $K_{1,t}$, with $t = \binom{k-1}{\ell-1} + 1$, in G that contains both x and y, and one of them as its center.

Unfortunately, this construction may require a lot of time, but it is still polynomial for fixed ℓ and k.

Theorem 18. *For all $2 \leqslant \ell < k$ there is a polynomial-time algorithm that decides whether a given graph is the ℓ-intersection graph of some k-uniform hypergraph H having all properties A, B, and $C(\ell, \binom{k-1}{\ell-1}(k-\ell) + 2k - \ell)$. If it is so, H is unique.*

Corollary 19. *Let $2 \leqslant \ell < k$ be integers. Let $p = p(n) \succ n^{-1/2+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, $\Omega_\ell \mathscr{H}_k(n, p)$ can be recognized in polynomial time.*

7. $(k-1)$-in-k graphs

For integers $2 \leqslant \ell < k$, the *ℓ-in-k graph* $\Phi_{\ell,k}(J)$ of a graph J has all complete ℓ-vertex subgraphs as vertices. Two such vertices are adjacent in $\Phi_{\ell,k}(J)$ if they are contained in some complete graph of at most k vertices. For $\ell = 2$ and $k = 3$ they are known under the name 'triangular line graph', and for $\ell = 2$, $k = 4$ as 'edge-clique graphs'; compare [14].

Note that $\Phi_{k-1,k}(J)$ is the underlying graph of $H^k(J)[k-1]$. Since, we know how to recognize $(k-1)$-set hypergraphs of K_k-hypergraphs of random graphs, our task for $\ell = k-1$ is essentially to reconstruct $H^k(J)[k-1]$ from $\Phi_{k-1,k}(J)$.

Now there are two kinds of cliques in $(k-1)$-in-k graphs $\Phi_{k-1,k}(J)$. For every complete graph with k vertices in J, its k $(k-1)$-element subsets correspond to some complete subgraph in $\Phi_{k-1,k}$ with k vertices. But this subgraph is actually a clique. We say this k-clique stems from some K_k in J. The subsets corresponding to every other clique must have all $k-2$ vertices in common. To be maximal means that we have a clique in the common neighborhood of $k-2$ pairwise adjacent vertices in J. If all cliques of J have more than $2k-2$ vertices, then all cliques of the second type have more than k vertices, and we can distinguish both types simply by their cardinality.

Proposition 20. *If $p \succ n^{-1/k+\varepsilon}$, for $\varepsilon > 0$, then almost no $\mathscr{G}(n, p)$ has a k-clique, i.e. a maximal complete subgraph with exactly k vertices.*

Proof. The probability that k fixed vertices form a k-clique is $p^{\binom{k}{2}}(1-p^k)^{n-k}$. These events are not pairwise independent, nevertheless the probability for a k-clique is at most

$$\begin{aligned}\mathscr{P}(\exists k\text{-clique}) &\leqslant \binom{n}{k} p^{\binom{k}{2}}(1-p^k)^{n-k} \\ &\leqslant \mathrm{O}(n^k)\left(1-\frac{n^{k\varepsilon}}{n}\right)^{n-k} \to 0 \quad \text{as } n\to\infty\end{aligned}$$

by Lemma 1. □

Therefore, for $(k-1)$-in-k graphs G of $J=\mathscr{G}(n,p)$ with $p \succ n^{-1/(2k-2)+\varepsilon}$, all k-cliques stem from K_ks in J. That is, we can reconstruct $H^k(J)[k-1]$ from $\Phi_{k-1,k}(J)$ by the following rule:

(5) Test whether the set of all k-cliques partitions the edge set of our graph G, and if it is so, choose the hypergraph T formed by all these k-cliques.

From this we can reconstruct $H^k(J)$ almost surely by Propositions 4 and 3. J is simply the underlying graph of H.

Theorem 21. *Let $k\geqslant 3$, and $p=p(n)\succ n^{-1/(2k-2)+\varepsilon}$, for $\varepsilon>0$. Then, almost surely, $\Phi_{k-1,k}(\mathscr{G}(n,p))$ can be recognized in polynomial time.*

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